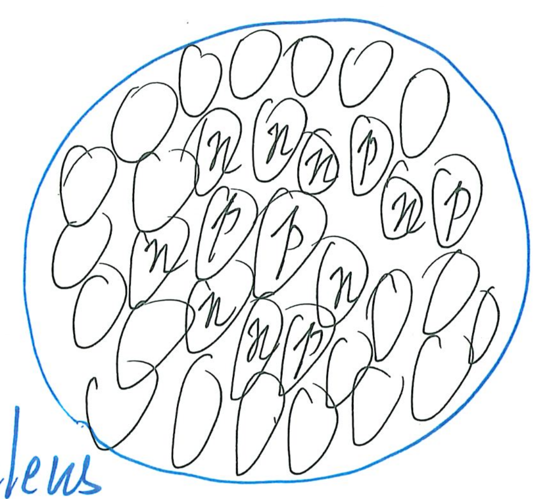
Tunneling and its Applications

- Traveling Waves in QM : Necessity and issues
- Continuity Equation and Probability Current density
- Tunneling
- Applications

Motivation (Phenomena)

- α - decay

neutrons + protons

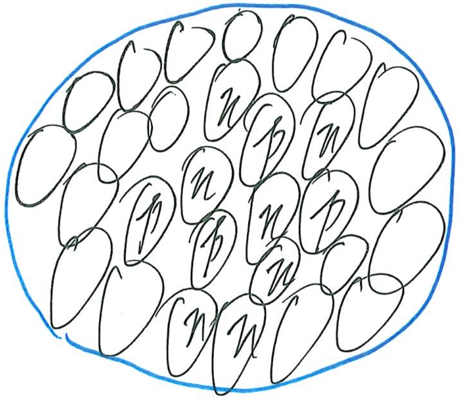


nucleus

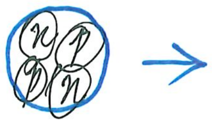
$\sim 10^{-14}\,m$

nucleons (p and n) bind by $\underbrace{\text{nuclear force}}$

only effective when nucleons are very close ($\sim 10^{-15}\,m$)



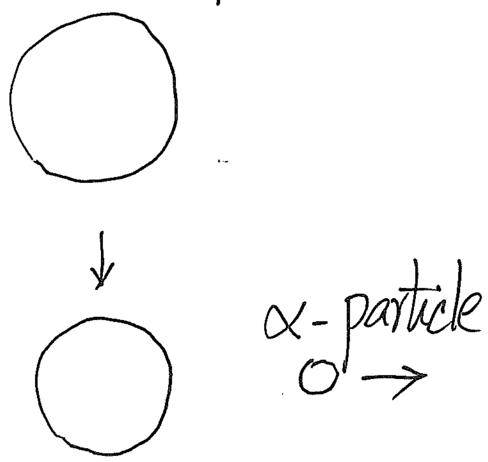
α - particle



[helium nucleus]

(2 protons + 2 neutrons)

Analogy



Nuclear Physics

Excited state —— "2"

Ground state - - - - "1"

↓ [spontaneous emission]

Excited state - - - - "2"

$\rightsquigarrow \hbar\omega$

Ground state —— "1"

Atomic Physics

$$\frac{dN(t)}{dt} = -\lambda N \leftarrow \text{(probabilistic)} \rightarrow \frac{dN_2(t)}{dt} = -\lambda_{21} N_2(t)$$

$N(t) = \#$ nuclei $\underbrace{\text{not decayed}}_{\text{"in excited state"}}$ at time t

$$\boxed{\therefore \text{ Life time } \tau = \frac{1}{\lambda}}$$

$$\boxed{\begin{array}{c}\text{half-life}\\ t_{1/2} \equiv \tau \cdot \ln 2\end{array}}$$

Data of 5 α-particle emitting nuclei

Nucleus	k.e. (MeV)	half-life $t_{1/2}$
^{216}Ra	9.5	0.18 μs
^{194}Po	7.0	0.7 s
^{240}Cm	6.4	27 days
^{226}Ra	4.9	1600 years
^{232}Th	4.1	14 billion years

α-particle k.e. don't vary much!

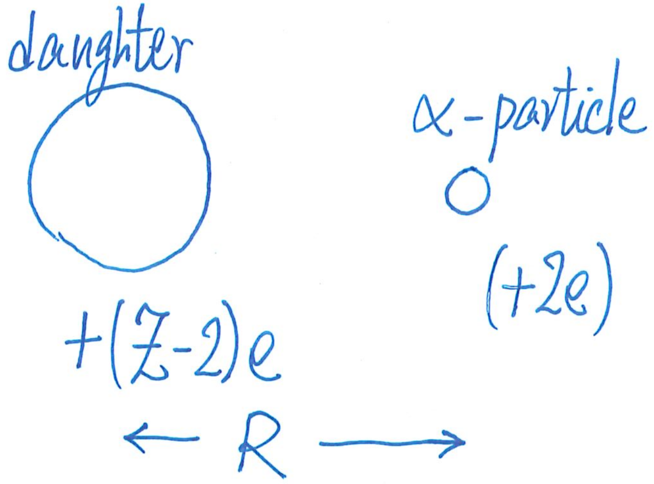
$t_{1/2}$ vary by a lot!

Many orders of magnitude

Note correlation:

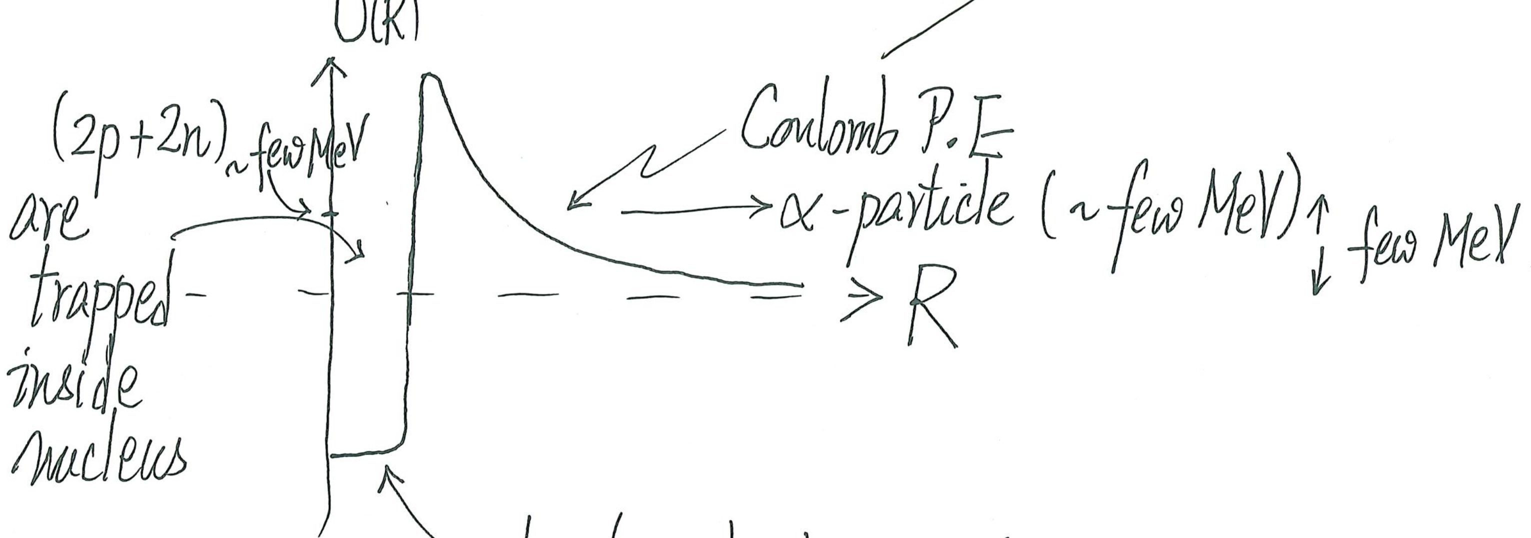
longer $t_{1/2}$ $\leftrightarrow$ smaller α-particle k.e.

shorter $t_{1/2}$ $\leftrightarrow$ higher α-particle k.e.



Coulomb repulsion

$$\left[P.E. = + \frac{(Z-2)\cdot 2e^2}{4\pi\varepsilon_0 R} \right]$$



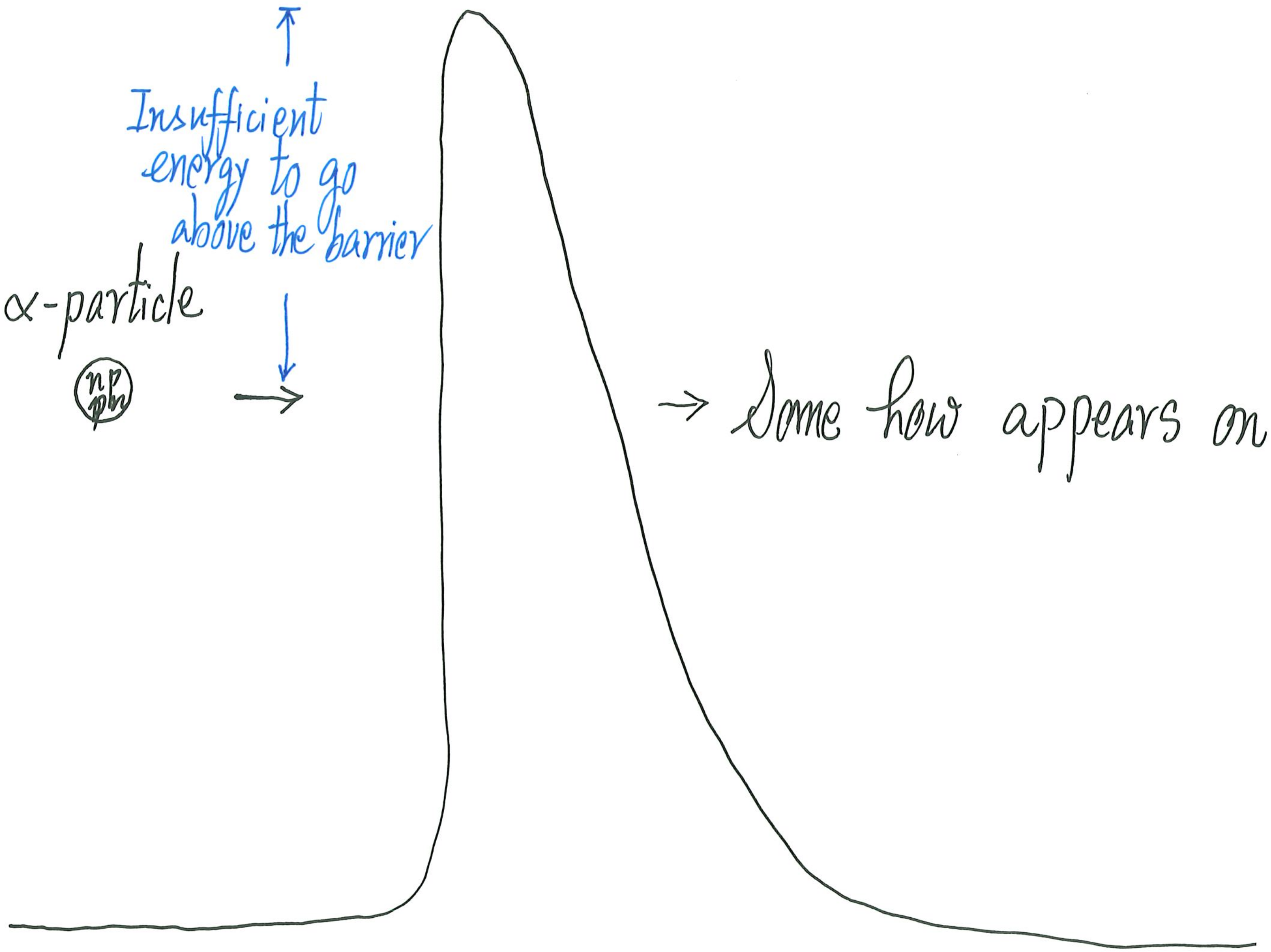
due to short-range ($\sim 10^{-15}$ m) nuclear force

Q: Does α-particle climb out of the barrier?

[No! α-particle k.e. is NOT that high!]

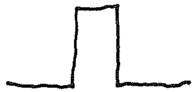
Q: How does α-decay occur? [Tunneling]

The situation is



To understand tunneling <u>properly</u>[+], we need some formal QM on:

- Traveling waves
- Continuity Equation and Probability Current Density

[+] Some books introduced tunneling in a way that students only know how to do  . Here, we will know how to do  ,  ,  , etc. at the end.

A. Describing something Traveling : Beyond Standing Waves

Looks like the "simplest" problem: Free Particle in 1D [$V(x)=0 \Rightarrow$ free]

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \qquad \text{(all } x\text{)}$$

$$\psi(x) \sim e^{+ikx}, \; e^{-ikx} \qquad \text{with } E(k) = \frac{\hbar^2 k^2}{2m}$$

$\overbrace{(+k \text{ and } -k}^{\text{degenerate}}$ have same E)

[no boundary, so no restriction on k]

$\psi_{+k}(x,t) \sim e^{ikx - i\omega(k)t}$ travels to the "right" (+ve x) as time evolves $\left(\omega(k) = \frac{E(k)}{\hbar}\right)$

$\psi_{-k}(x,t) \sim e^{-ikx - i\omega(k)t}$ travels to the "left" (−ve x) as time evolves

"Not-so-difficult" to have something traveling!? Is it?

A technical issue

- Can't normalize $\psi(x) = A e^{ikx}$ (OR e^{-ikx}) in usual way

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = |A|^2 \int_{-\infty}^{\infty} dx = |A|^2 \cdot \infty \quad ?$$

[This problem is "expected": e^{ikx} has a definite momentum $+\hbar k$]
$\Rightarrow \Delta x \to \infty$

Ways out

(a) Re-define normalization condition [invoke Dirac δ-function]

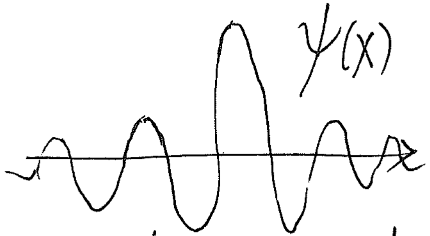
$$\int_{-\infty}^{\infty} \psi_{k'}^{*}(x)\, \psi_k(x)\, dx = \delta(k-k')$$

- has mathematics root in Fourier analysis
- but avoided normalization problem by adopting a different condition

(b) <u>Be strict and formal</u>: <u>Wave Packet can be normalized</u>

- QM wavefunctions must obey normalization condition
- Must use $\underbrace{\text{wave packet}}$ (not just $\sim e^{ikx}$)

form by superposition of e^{ikx} of different k's



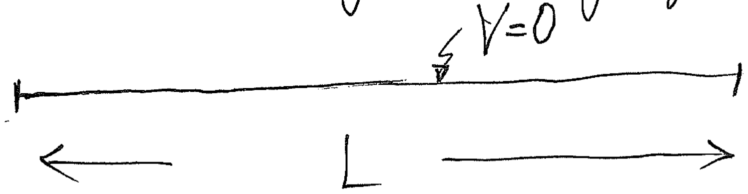
$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\phi(k)} e^{ikx} dk$$

Can be normalized!

different k-components to give $\psi(x)$

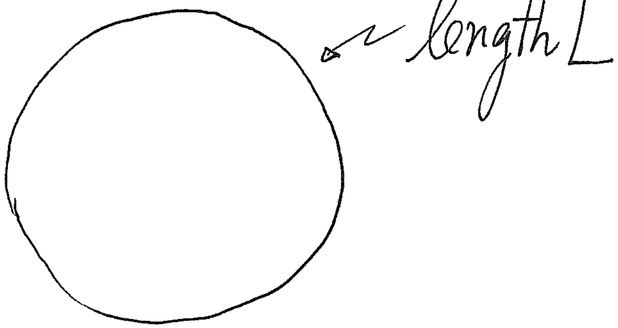
- This really solves the normalization problem
- But it makes calculations harder to do (but can be done)
- $\psi(x,t)$ sees wave packet moving and spreading

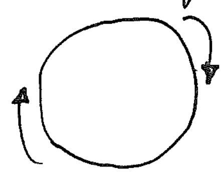
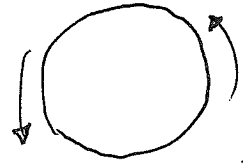
(c) Mimicing traveling freely in infinite space by a large finite space



"Particle-in-a-box" Again? No!

Periodic boundary condition: Hook up two ends



- traveling around ($\approx$ infinite system)
- e^{ikx} (going around )
- e^{-ikx} (going around )

$$\psi_k(x) = \frac{1}{\sqrt{L}} e^{ikx}$$ is normalized in ordinary sense

[always think that L is large]

This is an earlier question in a Problem Set

Buy one get one free: 1D travelling wave in a never-ending track of finite length

In 1D particle-in-a-box problem, we have $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$ inside the well where $U(x) = 0$ and thus the particle is free inside the well. We used standing wave $\psi(x) \sim \sin(kx)$ as the solutions. Everything is fine. In Problem Set 4, you worked out that for these energy eigenfunctions, $\langle p \rangle = 0$. This is right. These eigenfunctions, therefore, cannot be used to describe something that is moving, e.g. an electron moving along a metal wire.

The technique you used in Problem 5.2 for the $\Phi(\phi)$ function can be used to handle 1D free-particle problems with solutions that are **travelling**.

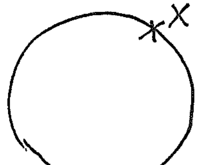
Consider a particle that is free ($U(x) = 0$) within a 1D system of length L. We want to mimic an infinite system by a finite L system. The trick is to **connect the two ends** together, i.e., what is called x is the same as $x + L$. The physical picture is that of a very long circular track (e.g. 100 km in circumference). As you run on the track, you will think that the track is a straight path (because it is very long) and it never ends (a circle never ends), although the path has a finite length. Technically, this is referred to be the **periodic boundary condition**. This is very important in solid state physics, as we want the electrons in a solid to move.

Solve the TISE with travelling waves of the form $\psi_k(x) \sim e^{ikx}$ and **find** the energy eigenvalue corresponding to $\psi_k(x)$. Requiring that $\psi_k(x)$ should be single-valued to be physically acceptable, **obtain** the allowed values of k and hence the allowed energy eigenvalues. An advantage of this approach is that $\psi_k(x)$ can be normalized. **Find** the normalization constant for $\psi_k(x)$.

$$\int_{-L/2}^{L/2}\left(\frac{1}{\sqrt{L}}e^{-ikx}\right)\left(\frac{1}{\sqrt{L}}e^{ikx}\right)dx = \frac{1}{L}\int_{-L/2}^{L/2}dx = \frac{1}{L}\cdot L = 1$$

$$\Rightarrow \boxed{\psi_k(x) = \frac{1}{\sqrt{L}}e^{ikx} \text{ is properly normalized}} \quad (1) \quad ☺$$

But there is a price to pay! ☹



x and $x+L$ refer to the same place

$$\psi_k(x+L) \overset{\text{[same place]}}{=} \psi(x)$$

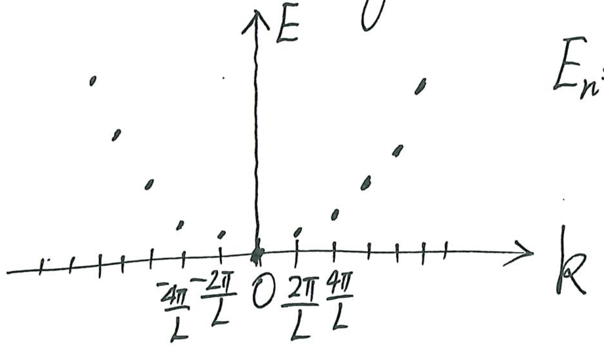
$$\Rightarrow e^{ikx}e^{ikL} = e^{ikx} \Rightarrow e^{ikL} = 1 \Rightarrow kL = 2\pi n \quad (n = 0, \pm 1, \pm 2, \cdots)$$

$$\Rightarrow \boxed{k_n = \frac{2\pi}{L}\cdot n} \quad (2) \quad (\because \text{There is a "boundary" set by } L)$$

k is discretized $\Rightarrow$ p $(\hbar k)$ is discretized $\Rightarrow$ $E = \dfrac{\hbar^2 k^2}{2m}$ is discretized

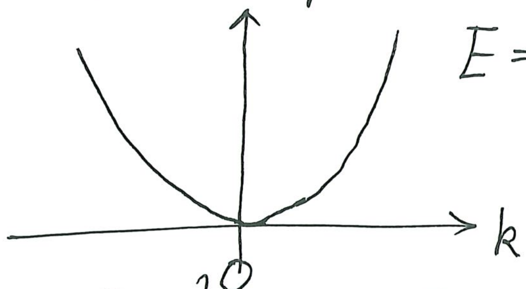
This is called "Box Normalization"

Box Normalization (length L)



$$E_n = \frac{\hbar^2 k_n^2}{2m}$$

Infinite Space



$$E = \frac{\hbar^2 k^2}{2m}$$

[continuous k]

But L is meant to be big

(Think $L \to \infty$) $\Rightarrow$ closely spaced k-values ($\approx$ continuum)

$$\psi_k(x) = \frac{1}{\sqrt{L}} e^{ikx} \quad \text{(normalized)}$$

$$k_n = n\frac{2\pi}{L} \quad (n = 0, \pm 1, \pm 2, \cdots)$$

- Retains ideas on QM wavefunctions learned so far & has traveling waves

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

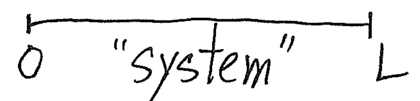
[δ-function normalization]

(any k)

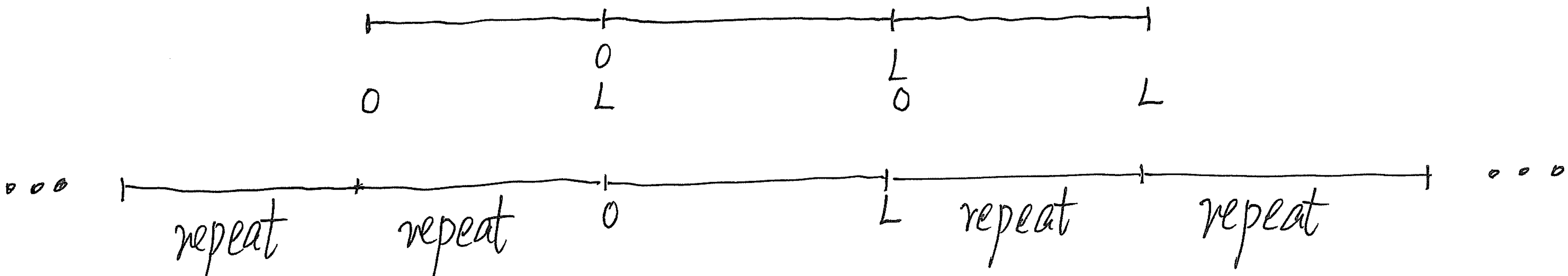
- Needs new way for "normalization" that works for continuous spectrum (eigenvalues) & has traveling waves

$$\psi_k(x) = \frac{1}{\sqrt{L}} e^{ikx} \quad ; \quad \underbrace{k_n = n\frac{2\pi}{L}}_{\text{finite } L \text{ (finite "Box") leads to discrete } k\text{'s}} \quad (n = 0, \pm 1, \pm 2, \cdots) \qquad (1)\&(2)$$

- The bigger the box, the less noticeable is the discreteness of k
- Another way to visualize box normalization & Periodic boundary condition

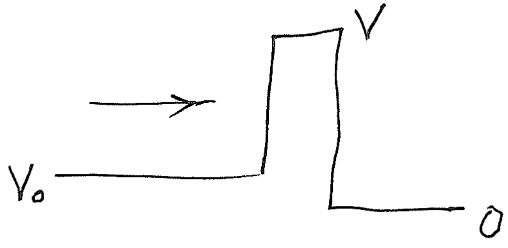


copy it and paste it to two sides



Mimicing an infinite system by repeating a finite system!

- Why do we want/need traveling waves?
  - incident wave upon a barrier in <u>tunneling problem</u>

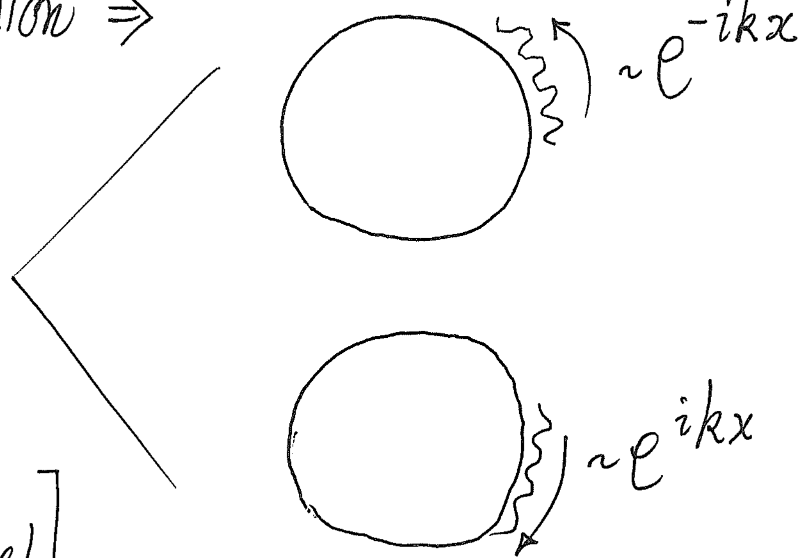


and other <u>scattering problems</u>

  - electrons travel in a metal to make it a conductor

metal wire

Think "box normalization" $\Rightarrow$



for an allowed k

Get ready for <u>Quantum Theory of Solid States</u>
[conduction when an $\vec{\mathcal{E}}$ field is applied]

$$\boxed{\text{Box Normalization is often used in Solid State Physics}}$$

Key Points

- We need traveling waves for some problems in QM
- Allowed energies form continuum spectrum: Wavefunctions cannot be normalized in usual way
- There are ways out
  - change normalization condition [δ-function]
  - form wave packets
  - Box normalization [comes with discrete energies that go continuum for large box]
- Need to be careful in treating traveling waves in QM, but there are ways out
- Get ready for solid state physics